Approximate Solution of the Effective Mass Klein-Gordon Equation for the Hulthen Potential with Any Angular Momentum

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Abstract The radial part of the effective mass Klein-Gordon equation for the Hulthen potential is solved by making an approximation to the centrifugal potential. The Nikiforov-Uvarov method is used in the calculations. Energy spectra and the corresponding eigenfunctions are computed. Results are also given for the case of constant mass.

1 Introduction

There has been a continues interest on the solutions of the Klein-Gordon (KG) equation and Dirac equation for some certain potentials. The KG equation is solved by assuming that the scalar potential equals or not equals to the vector potential, such as Rosen-Morse-type potentials, Morse-like potential [1–10]. On the other hand, it is also solved for the case of mixing of scalar and vector potentials, such as vector-scalar Coulomb, kink-like potentials, harmonic oscillator, Hartman potential, and Hulthen-type potential [11–17]. Further, the Dirac equation is solved for different types of potentials, such as harmonic, and linear potentials [18], for an uniform magnetic field [19], generalized asymmetrical Hartmann potentials [20]. Various methods are used in the solutions, such as Nikiforov-Uvarov (NU) method, by using the hypergeometric type equations, and separation of variables [21–26].

In recent years, the effects of the position-dependent the mass on the energy spectra and corresponding eigenvalues of the above equations has been received a great attention [27, 28]. The exact solutions of the above equations, and the Schrödinger equation in

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the context of spatially dependent mass have been studied by many authors [29–35]. Many different types of mass distributions have been used in literature, such as polynomial mass function, exponential, and hyperbolic mass distributions [39]. In the present work, we prefer an exponential type mass function to find out the energy spectrum and corresponding wave functions of the Hulthen potential. The Hulthen potential has applications in the wide range of different areas such as nuclear, and particle physics, atomic physics, solid state, and chemical physics [9].

In this study, we solve the radial KG equation for the Hulthen potential by using NU-method within the framework of an approximation to the centrifugal potential. The NU-method is based on solving the second order equation by reducing to a generalized equation of hypergeometric type [40].

The organization of this work is as follows. In Sect. 2, we solve the radial part of the KG equation by using an approximation for the centrifugal term [42, 43], and compute energy eigenvalues of the bound states and the corresponding eigenfunctions. We give our conclusions in Sect. 3.

2 Nikiforov-Uvarov Method and Bound States

The radial part of the KG equation reads [36–38]

$$\left\{\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{1}{\hbar^2 c^2}[m^2 c^4 - (E-V)^2]\right\}\phi(r) = 0 \quad (0 \leq r \leq \infty), \qquad (1)$$

where ℓ is the angular-momentum quantum number, E is the energy of the particle, c is the velocity of the light.

The Hulthen potential has the form [41]

$$V(r) = -V_0 \frac{e^{-\beta r}}{1 - e^{-\beta r}}, \qquad (2)$$

where V_0, and $\beta \equiv 1/a$ are constant parameters.

Equation (1) can not be solved exactly because of the centrifugal potential. So, one has to use an approximation for this term. This approximation can be taken as [42, 43]

$$\frac{\ell(\ell+1)}{r^2} \simeq \beta^2 \ell(\ell+1) \frac{e^{-\beta r}}{(1 - e^{-\beta r})^2}. \qquad (3)$$

There are very different mass distributions used in the literature, such as an exponentially, and quadratically mass functions [29], trigonometric mass-distributions, and mass functions of the forms $m(r) = r^\alpha$, especially for three-dimensional problems [32, 33]. Here, we prefer to use the following spatially dependent mass

$$m(r) = m_0 - \frac{m_1}{1 - e^{-\beta r}} \quad (m_0 > m_1). \qquad (4)$$

which corresponds to a decay particle, and provides us an approximate solution of the radial part of the effective KG equation. m_0 and m_1 in this distribution are two arbitrary positive parameters. The mass function of that form enable us to check out the results in the limit of the constant mass.

Substituting (2) and (4) into (1), and by using (3), we get

$$\left\{ \frac{d^2}{dr^2} - \beta^2 \ell(\ell+1) \frac{e^{-\beta r}}{(1-e^{-\beta r})^2} + \frac{1}{\hbar^2 c^2} \left[E^2 + 2EV_0 \frac{e^{-\beta r}}{1-e^{-\beta r}} + V_0^2 \frac{e^{-2\beta r}}{(1-e^{-\beta r})^2} \right. \right.$$

$$\left. \left. - m_0^2 c^4 + \frac{2m_0 m_1 c^4}{1-e^{-\beta r}} - \frac{m_1^2 c^4}{(1-e^{-\beta r})^2} \right] \right\} \phi(r) = 0. \tag{5}$$

By using the transformation $z = 1 - e^{-\beta r}$ $(0 \leq z \leq 1)$, we obtain

$$\frac{d^2\phi(z)}{dz^2} - \frac{z}{z(1-z)} \frac{d\phi(z)}{dz} + \frac{1}{[z(1-z)]^2} \left[-a_1^2 z^2 - a_2^2 z - a_3^2 \right] \phi(z) = 0, \tag{6}$$

where

$$-a_1^2 = Q^2 (E^2 - 2EV_0 + V_0^2 - m_0^2 c^4),$$
$$-a_2^2 = Q^2 (2m_0 m_1 c^4 + 2EV_0 - 2V_0^2) + \ell(\ell+1), \tag{7}$$
$$-a_3^2 = Q^2 (V_0^2 - m_1^2 c^4) - \ell(\ell+1)$$

and $Q^2 = 1/\beta^2 \hbar^2 c^2$. Now to apply the NU-method [40], we rewrite (6) in the following form

$$\phi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} \phi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} \phi(z) = 0, \tag{8}$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials with second-degree, at most, and $\tilde{\tau}(z)$ is a polynomial with first-degree. We define a transformation for the total wave function as

$$\phi(z) = \xi(z)\psi(z). \tag{9}$$

Thus (8) is reduced to a hypergeometric type equation

$$\sigma(z)\psi''(z) + \tau(z)\psi'(z) + \lambda \psi(z) = 0. \tag{10}$$

We also define the new eigenvalue for the (8) as

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2} \sigma'' \quad (n = 0, 1, 2, \ldots) \tag{11}$$

where

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z). \tag{12}$$

The derivative of $\tau(z)$ must be negative. $\lambda(\lambda_n)$ is obtained from a particular solution of the polynomial $\psi_n(z)$ with the degree of n. $\psi_n(z)$ is the hypergeometric type function whose solutions are given by [40]

$$\psi_n(z) = \frac{b_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \tag{13}$$

where the weight function $\rho(z)$ satisfies the equation

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z). \tag{14}$$

On the other hand, the function $\xi(z)$ satisfies the relation

$$\xi'(z)/\xi(z) = \pi(z)/\sigma(z). \tag{15}$$

Comparing (6) with (8), we have

$$\tilde{\tau}(z) = -z, \qquad \sigma(z) = z(1-z), \qquad \tilde{\sigma}(z) = -a_1^2 z^2 - a_2^2 z - a_3^2. \tag{16}$$

The $\pi(z)$ has the form

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}, \tag{17}$$

or, explicitly

$$\pi(z) = \frac{1}{2}(1-z) \mp \sqrt{\left(\frac{1}{4} + a_1^2 - k\right)z^2 + \left(a_2^2 + k\frac{1}{2}\right)z + a_3^2 + \frac{1}{4}}. \tag{18}$$

The constant k is determined by imposing a condition such that the discriminant under the square root should be zero. The roots of k are $k_{1,2} = -a_2^2 - 2a_3^2 \mp A\sqrt{1 + 4a_3^2}$, where $A = \sqrt{a_3^2 + a_2^2 + a_1^2}$. Substituting these values into (18), we get for $\pi(z)$

$$\pi(z)(k \to k_1) = \frac{1}{2}(1-z) \mp \left[\left(A - \sqrt{\frac{1}{4} + a_3^2}\right)z + \sqrt{\frac{1}{4} + a_3^2}\right], \tag{19}$$

and

$$\pi(z)(k \to k_2) = \frac{1}{2}(1-z) \mp \left[\left(A + \sqrt{\frac{1}{4} + a_3^2}\right)z - \sqrt{\frac{1}{4} + a_3^2}\right]. \tag{20}$$

Now we calculate the polynomial $\tau(z)$ from $\pi(z)$ such that its derivative with respect to z must be negative. Thus we take the first choice

$$\tau(z) = 1 - 2\sqrt{\frac{1}{4} + a_3^2} - 2\left(A - \sqrt{\frac{1}{4} + a_3^2} + 1\right)z. \tag{21}$$

The constant $\lambda = k + \pi'(z)$ becomes

$$\lambda = -a_2^2 - 2a_3^2 + A\sqrt{1 + 4a_3^2} - \frac{1}{2} - \left(A - \sqrt{\frac{1}{4} + a_3^2}\right), \tag{22}$$

and (11) gives us

$$\lambda_n = 2n\left(A - \sqrt{\frac{1}{4} + a_3^2} + 1\right) + n^2 - n. \tag{23}$$

Substituting the values of the parameters given by (7), and setting $\lambda = \lambda_n$, one can find the energy eigenvalues as

$$
\begin{aligned}
E_{n\ell} = \frac{V_0}{2} + \frac{1}{4Q^2(N^2 + 4Q^2V_0^2)} &\big\{ 8Q^3 m_1 c^4 (m_1 - 2m_0) \\
&\mp N \big[16Q^4 (V_0^2 - m_1^2 c^4)(m_1^2 c^4 - 4m_0 m_1 c^4 + 4m_0^2 c^4 - V_0^2) \\
&+ 8Q^2 N^2 (2m_0^2 c^4 - 2m_0 m_1 c^4 + m_1^2 c^4 - V_0^2) - N^4 \big]^{1/2} \big\},
\end{aligned}
\tag{24}
$$

where

$$
N = (2n+1) + \sqrt{1 + 4a_3^2}.
\tag{25}
$$

We see that the energy levels for particles and antiparticles are symmetric, and the ground state energy is different from zero.

We also get the energy eigenfunctions for the constant mass case for s-states

$$
E_{n,\ell=0}^{m_1=0} = \frac{V_0}{2} \pm N' \sqrt{\frac{m_0^2 c^4}{4Q^2 V_0^2 + N'^2} - \frac{1}{16Q^2}},
\tag{26}
$$

where

$$
N' = (2n+1) + \sqrt{1 - 4Q^2 V_0^2}.
\tag{27}
$$

It is the same result with those in literature for $q = 1$ [25, 26].

Now let us find the eigenfunctions. We first compute the weight function from (12) and (14)

$$
\rho(z) = z^{\sqrt{1+4a_3^2}}(1-z)^{2A}
\tag{28}
$$

and the wave functions become

$$
\psi_n(z) = \frac{b_n}{z^{\sqrt{1+4a_3^2}}(1-z)^{2A}} \frac{d^n}{dz^n} \left[z^{n+\sqrt{1+4a_3^2}}(1-z)^{n+2A} \right]
\tag{29}
$$

where b_n is a normalization constant. The polynomial solutions can be written in terms of the Jacobi polynomials [44, 45]

$$
\psi_n(z) = b_n P_n^{(\sqrt{1+4a_3^2},\, 2A)}(1-2z), \quad 2A > -1, \ \sqrt{1+4a_3^2} > -1.
\tag{30}
$$

On the other hand, the other part of the wave function is obtained from (15) as

$$
\xi(z) = z^{\frac{1}{2}[1+\sqrt{1+4a_3^2}]}(1-z)^A.
\tag{31}
$$

Thus, the total eigenfunctions take

$$
\phi_n(z) = b_n' (1-z)^A z^{\frac{1}{2}[1+\sqrt{1+4a_3^2}]} P_n^{(\sqrt{1+4a_3^2},\, 2A)}(1-2z)
\tag{32}
$$

where b_n' is the new normalization constant.

Total wave function for the constant mass case for any ℓ-state takes

$$\phi_{n\ell}^{m_1=0}(z) = b_n''(1-z)^{A'} z^{\frac{1}{2}[1+\sqrt{1+4a_3'^2}]} P_n^{(\sqrt{1+4a_3'^2}, 2A')}(1-2z). \tag{33}$$

It is consistent with the results obtained in the literature [25, 26]. The parameters for this case are given by $A' = \sqrt{a_3'^2 + a_2'^2 + a_1'^2}$, and

$$\begin{aligned}
a_1'^2 &= a_1^2, \\
a_2'^2 &= 2Q^2 V_0(V_0 - E) - \ell(\ell+1), \\
a_3'^2 &= -Q^2 V_0^2 + \ell(\ell+1).
\end{aligned} \tag{34}$$

3 Conclusion

We have obtained approximate solution of the radial part of the effective mass KG equation for the Hulthen potential in the framework of an approximation to the centrifugal term for any ℓ values. We have obtained the energy spectra and the corresponding radial part of the wave functions by applying the NU-method. We have found a real energy spectra for the Hulthen potential in the PDM case. To check our results, we have also calculated the energy eigenvalues and eigenfunctions of the particle and antiparticles for the case of constant mass limit for s-waves, and seen that the results are consistent with those in the literature [25, 26].

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